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On a Holomorphic Family of Stein Manifolds with Strongly Pseudoconvex Boundaries

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Abstract
We study the stable embedding problem for a CR family of 3-dimensional strongly pseudoconvex CR manifolds with each fiber bounding a stein manifold.

Keywords CR manifold · Stable embedding · L^2 method

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1 Introduction

Let X be a compact strongly pseudoconvex CR manifold. The question of whether or not X admits a CR embedding into a complex Euclidean space has attracted a lot attention. This amounts to showing that the manifold has a sufficiently rich collection of global CR functions. It was shown by Boutet de Monvel [4] that the answer is affirmative if the dimension of X is at least five. In contrast, if X has dimension three, X may not be even locally embeddable, see [21,22,27]. Furthermore, there are examples [5,14,28] which show that even when the CR structure on X is locally embeddable (for example, when it is real analytic), it can happen that the global CR functions on X fail to separate points of X. It was shown in [6] that, in a rather precise sense, "generic" perturbations of the standard CR structure on the three sphere are non-embeddable.

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On the other hand, if a compact three-dimensional strongly pseudoconvex CR manifold admits a transversal CR S^1-action, it was shown by Lempert [24], Epstein [10], and recently in [16,17] by using the Szegő kernel that such CR manifolds can always be CR embedded into a complex Euclidean space.

In recent years, much progress has been made in understanding the embedding question from a deformational point of view, that is, for CR structures which lie in a small neighborhood of a fixed embedded structure, see e. g. [6,10,11,18,23–26,30].

Problem 1.1 ([25]) *Suppose $f : (X, HX, J) \to \mathbb{C}^k$ is a CR embedding, and let (X, HX, J') be another CR manifold with J' close to J. Assuming (X, HX, J') is CR embeddable into some $\mathbb{C}^l$, does it follow that it also admits a CR embedding $f' : (X, HX, J') \to \mathbb{C}^k$ with f' close to f?*

If this holds for J' close to J, we say that f is a stable embedding. We say that two tensors are close if they are close in the C^∞ topology on the appropriate space. The problem of stable embedding was first studied by Tanaka [29], who showed that for a smooth family of compact strictly pseudoconvex CR manifolds of dimension at least five, any embedding is stable provided the dimension of first Kohn–Rossi cohomology groups of the fibers do not depend on the parameter.

However, for three-dimensional case Catlin and Lempert [8] constructed an example to show that the unstable embedding exists. They constructed a family of unit circle bundle over a fixed compact Riemann surface. The instability of CR embedding of the unit circle bundles is a consequence of the instability of the very ample line bundles. But if a strictly pseudoconvex CR manifold X admits an embedding in $\mathbb{C}^2$, Lempert [25] showed that this embedding is stable. In general, Lempert proposed the following

Conjecture 1.2 ([25]) *Let (X, HX, J) be a three-dimensional strongly pseudoconvex CR manifold. If (X, HX, J) is the boundary of a stein manifold, then any CR embedding $f : (X, HX, J) \to \mathbb{C}^k$ is stable.*

Huang, Luk, and Yau [18] studied the stability of embedding for a CR family of strongly pseudoconvex CR manifolds with the CR structures of the fibers CR depending on the parameters. In [18], the dimension of each fiber has to be greater or equal to five. For a CR family of three-dimensional CR manifolds, the problem of stability of embedding is still open. The CR dependence on the parameters for the CR families is crucial for the studies in the deformation theory of the complex structure of isolated singularities. Here, we refer the readers to [7,18,19] and the references therein.

In this paper, we will continue the program which was started in [18] on the stability of embedding problems for a CR family of strongly pseudoconvex CR manifolds. First, we recall the notations in [18, Definition 1.1].

Definition 1.3 Let $\Delta = \{t \in \mathbb{C} : |t| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\{X_t\}_{t\in\Delta}$ be a parameterized family of compact strongly pseudoconvex CR manifolds of real dimension $2n - 1$. The family is said to be a CR family, or X_{t_1} is said to be a CR deformation of X_{t_2} for any $t_1, t_2 \in \Delta$ if there is a strongly pseudoconvex CR manifold $\mathcal{X}$ and a C^∞ CR map $\pi : \mathcal{X} \to \Delta$ such that (I) π is a proper submersion; (II) for any $t \in \Delta$, $X_t = \pi^{-1}(t)$ and X_t is a CR submanifold of $\mathcal{X}$.

If $n \geq 3$, that is, the dimension of each fiber is at least five, Huang–Luk–Yau [18] established the stability of embedding for a CR family of strongly pseudoconvex CR manifolds under the condition that the dimension of the first Kohn–Rossi cohomology of each fiber does not depend on the parameter. One key step in their work is the simultaneous filling of the CR family by a holomorphic family of convex–concave complex manifolds. The argument in [18] does not work on the case when the dimension of each fiber is three. An open problem was stated in [19].

Problem 1.4 ([19]) *Let $\{X_t\}_{t\in\Delta}$ be a CR family of 3-dimensional strongly pseudoconvex CR manifold. Suppose that the total space admits a normal stein filling. Suppose that X_0 is embedded into some $\mathbb{C}^N$. Under what conditions, can the nearby X_t be CR embedded into the same $\mathbb{C}^N$?*

In order to study the Conjecture 1.2, we assume that X_0 can be filled by a stein manifold M_0. Furthermore, we consider a special case of Problem 1.4. We assume that the nearby X_t can be simultaneously filled by a complex manifold M_t with X_t as its strongly pseudoconvex boundary and as a consequence we will have a holomorphic family of complex manifolds with strongly pseudoconvex boundaries. Here, when we say a complex manifold $\overline{M}$ with smooth boundary X, we mean that $\overline{M}$ has a cover by coordinate patches $\{U_\alpha\}$ with C^∞ coordinates $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha : U_\alpha \cap M \to \mathbb{C}^n$ is holomorphic. Then X inherits a natural CR structure from M. We may always assume that M is a relatively compact open subset of some C^∞ manifold M' and X is a smooth submanifold of M'.

Definition 1.5 Let $\Delta = \{t \in \mathbb{C} : |t| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\{\overline{M_t}\}_{t\in\Delta}$ be a parameterized family of complex manifolds with smooth boundaries. The family is said to be a holomorphic family if there is a complex manifold $\overline{\mathcal{M}}$ (with smooth boundary) and a smooth map $\pi : \overline{\mathcal{M}} \to \Delta$ such that (I) π is a proper submersion; (II) The restriction of π on $\mathcal{M}$ is holomorphic; (III) for any $t \in \Delta$, $\overline{M_t} = \pi^{-1}(t)$ is a complex submanifold of $\overline{\mathcal{M}}$.

In what follows, we denote by $(\overline{\mathcal{M}_\varepsilon}, \Delta_\varepsilon, \pi)$ a holomorphic family of complex manifolds with smooth boundaries where $\Delta_\varepsilon := \{t \in \mathbb{C} : |t| < \varepsilon\}$. Suppose that M_0 is a stein manifold with strongly pseudoconvex boundary.

We denote by $X_t = \partial M_t$ the boundary of M_t for any $t \in \Delta_\varepsilon$ and set $\mathcal{X}_\varepsilon = \cup_{t\in\Delta_\varepsilon} X_t$. The boundary of $\mathcal{M}_\varepsilon$ has two pieces Y_0 and Y_1, where $Y_0 = \mathcal{X}_\varepsilon$ and $Y_1 = \pi^{-1}\{|t| = \varepsilon\}$. In what follows, we assume that $\mathcal{M}_\varepsilon$ is contained in a large differential manifold $\mathcal{M}'$ with Y_0 and Y_1 smooth submanifolds of $\mathcal{M}'$. Then we can state our main result.

Theorem 1.6 *Let $(\overline{\mathcal{M}_\varepsilon}, \Delta_\varepsilon, \pi)$ be a holomorphic family of complex manifolds with each fiber of complex dimension 2. Assume that Y_0 is strongly pseudoconvex with respect to $\mathcal{M}_\varepsilon$ and M_0 is a stein manifold. Let $\overline{M_t} = \pi^{-1}(t)$ and $X_t = \partial M_t$, $\forall t \in \Delta_\varepsilon$. If X_0 can be CR embedded into $\mathbb{C}^m$ for some m by a CR map $F_0 : X_0 \to \mathbb{C}^m$, then there is a CR embedding $G : \mathcal{X}_\varepsilon \to \mathbb{C}^{m+1}$ when ε is sufficiently small such that $G|_{X_t}$ CR embeds X_t into $\mathbb{C}^m \times \{t\}$ and $G|_{X_0} = (F_0, 0)$.*

2 L^2-Method for the $\overline{\partial}$-Equation on $\mathcal{M}_\varepsilon$

We now proceed to study the stability problem for a CR family of strongly pseudoconvex CR manifolds which bound a holomorphic family of complex manifolds. For this, we need study the $\overline{\partial}$-equation on $\mathcal{M}_\varepsilon$. However, the non-smooth boundary of $\mathcal{M}_\varepsilon$ makes a direct approach difficult. Thus, we need find a Hermitian metric on $\mathcal{M}_\varepsilon$ which can blow up Y_0 to infinity.

We now introduce a Hermitian metric ds^2 over $\mathcal{M}_\varepsilon$ such that the following properties hold: (a) First, ds^2 is smooth up to $\overline{\mathcal{M}_\varepsilon} \setminus Y_1$; (b) we can find a finite covering $\{U_\alpha\}_\alpha$ of $\overline{\mathcal{M}_\varepsilon}$ and coordinates (z_α, t) on each U_α such that z_α is smooth on U_α and $z_\alpha|_{\mathcal{M}_\varepsilon \cap U_\alpha}$ is holomorphic. For convenience, we will omit α in the notation z_α. With respect to the coordinates (z, t) with $z = (z_1, z_2)$,

$$ds^2 = \sum_{j,k=0}^{2} h_{j\bar{k}}(z, t) dz_j \otimes d\bar{z}_k + \frac{1}{(\varepsilon^2 - |t|^2)^2} dt \otimes d\bar{t},$$

where $z_0 = t$ and $h_{j\bar{k}} \in C^\infty(U_\alpha \cap \overline{\mathcal{M}_\varepsilon})$ when U_α intersects with the boundary of $\mathcal{M}_\varepsilon$. Write $\eta(t) = -\log(\varepsilon^2 - |t|^2)$, $e^{2\eta(t)} = \frac{1}{(\varepsilon^2 - |t|^2)^2}$. Then the volume form on $\mathcal{M}_\varepsilon$ with respect to ds^2 is given by $dv = h_0(z, t) e^{2\eta(t)} d_{Eucl}$, where $h_0(z, t) \in C^\infty(U_\alpha \cap \overline{\mathcal{M}_\varepsilon})$ and d_{Eucl} is the standard volume form on $\mathbb{C}^3$.

Lemma 2.1 *For sufficiently small ε, there exists a strictly plurisubharmonic function φ on $\mathcal{M}_\varepsilon$ which is smooth up to the boundary $\partial\mathcal{M}_\varepsilon$. As a consequence, for $t \in \Delta_\varepsilon$ each M_t is a stein manifold with strongly pseudoconvex boundary.*

Proof By the assumption of Theorem 1.6, M_0 is a stein manifold with a strongly pseudoconvex boundary. By Theorem 4.1 in [15], there is a large stein manifold Y which contains M_0 as an open subset. Then M_0 can be embedded to some $\mathbb{C}^N$ by a holomorphic map F which is smooth up to the boundary. Then $\varphi_0 = F^*(\sum_{j=1}^N |z_j|^2)$ is a strictly plurisubharmonic function on M_0 and φ_0 is smooth up to the boundary of M_0. Let $f : M_0 \times \Delta_\varepsilon \to \mathcal{M}_\varepsilon$ be a diffeomorphism which is smooth up to the boundary satisfying $f|_{M_0 \times \{0\}} = id$. Let $p_r : M_0 \times \Delta_\varepsilon \to M_0$ be the natural projection. Take $\varphi = \varphi_0 \circ p_r \circ f^{-1} + l|t \circ \pi|^2$ and l is a positive number. Then φ will be a strictly plurisubharmonic function on $\mathcal{M}_\varepsilon$ which is smooth up to the boundary of $\mathcal{M}_\varepsilon$ when l is sufficiently large and ε is sufficiently small. Thus, each M_t is a complex manifold without compact positive dimensional subvariety. Since each M_t has a strongly pseudoconvex boundary, then by a result of Grauert [13] (also see [2, Corollary, p. 233]) we have that each M_t is a stein manifold. □

Lemma 2.1 implies that the holomorphic family of complex manifolds in Theorem 1.6 is actually a holomorphic family of stein manifolds.

In what follows, we will fix a smooth real-valued function r over $\overline{\mathcal{M}_\varepsilon}$ such that r is a defining function of Y_0. For $\tau \gg 1$, write

$$\Lambda_{\tau,k} = -\tau \log(\varepsilon^2 - |t|^2) + k\varphi$$

which is a strongly plurisubharmonic function on $\mathcal{M}_\varepsilon$ when τ is sufficient large. We will use $\Lambda_{\tau,k}$ as a weight function to solve the $\overline{\partial}$-equation on $\mathcal{M}_\varepsilon$.

We will work with the following orthonormal basis over U_α:

$$\omega^0 = \frac{b_0}{\varepsilon^2 - |t|^2} dt, \omega^j = \sum_{k=1}^{2} b_k^j dz_k + b_j dt, \; j = 1, 2, \tag{2.1}$$

where $b_0, b_j, b_k^j \in C^\infty(U_\alpha \cap \overline{\mathcal{M}_\varepsilon})$ and $b_0 > 0$. Let $\{L_j\}_{j=0}^2$ be the frame over U_α dual to $\{\omega^j\}_{j=0}^2$ in (2.1). Then

$$L_0 = (\varepsilon^2 - |t|^2)\left(a_0 \frac{\partial}{\partial t} + \sum_{k=1}^{2} a_k \frac{\partial}{\partial z_k}\right), L_j = \sum_{k=1}^{2} a_j^k \frac{\partial}{\partial z_k}, \; j \neq 0. \tag{2.2}$$

Here, $a_0, a_k, a_j^k \in C^\infty(U_\alpha \cap \overline{\mathcal{M}_\varepsilon})$.

Let $\Omega^{0,q}(\overline{\mathcal{M}_\varepsilon})$ be the space of $(0, q)$-forms on $\mathcal{M}_\varepsilon$ which are smooth up to the boundary of $\overline{\mathcal{M}_\varepsilon}$. Let $\Omega_c^{0,q}(\mathcal{M}_\varepsilon)$ be the subspace of $\Omega^{0,q}(\overline{\mathcal{M}_\varepsilon})$ with the elements having compact support in the interior of $\mathcal{M}_\varepsilon$. A smooth $(0, q)$-form $u \in \Omega^{0,q}(\overline{\mathcal{M}_\varepsilon})$ is said to have compact support along t-direction if it vanishes when $\varepsilon - |t| < c_u$ with c_u a sufficiently small constant. We denote by $L^2_{(0,q)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$ the completion of $\Omega_c^{0,q}(\mathcal{M}_\varepsilon)$ under the $\Lambda_{\tau,k}$-weighted L^2-norm.

Consider the $\overline{\partial}$-operator as a maximally closed extended operator acting on a dense subspace of the $\Lambda_{\tau,k}$-weighted L^2-space of functions, $(0, 1)$-forms. Let $\overline{\partial}^*_{\Lambda_{\tau,k}}$ be its Hilbert adjoint from the space of $(0, 1)$-forms into the space of functions.

Lemma 2.2 *For all $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$, there exists a sequence $\{u_m\}$ which belong to $\Omega^{0,1}(\overline{\mathcal{M}_\varepsilon})$ and have compact supports along t-direction such that as $m \to \infty$,*

$$\|u_m - u\|_{\Lambda_{\tau,k}} + \|\overline{\partial} u_m - \overline{\partial} u\|_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u_m - \overline{\partial}^*_{\Lambda_{\tau,k}} u\|_{\Lambda_{\tau,k}} \to 0. \tag{2.3}$$

Proof For any $v \in \mathbb{N}$, choose $\eta_v \in C_0^\infty(\Delta_\varepsilon)$ such that $\eta_v(t) \equiv 1$ when $|t| < \varepsilon - \frac{1}{v}$, $\eta_v(t) \equiv 0$ when $|t| > \varepsilon - \frac{1}{2v}$. Then $|D\eta_v| \leq Cv$ and the point wise norm $|\overline{\partial}\eta_v|_{ds^2} \leq C_0$ for some constants C, C_0 independent of v. Hence, for any $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$, $\overline{\partial}(\eta_v u) \to \overline{\partial} u$ and $\overline{\partial}^*_{\Lambda_{\tau,k}}(\eta_v u) \to \overline{\partial}^*_{\Lambda_{\tau,k}} u$ in the respected $\Lambda_{\tau,k}$-weighted L^2-norms. Now applying the Friedrich–Hörmander approximation theorem, we see that for any $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$, there is a sequence $\{u_m\} \subset \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}})$, each of which is compactly supported along t-direction and is smooth up to the boundary of $\overline{\mathcal{M}_\varepsilon}$ such that

$$u_m \to u, \overline{\partial} u_m \to \overline{\partial} u, \overline{\partial}^*_{\Lambda_{\tau,k}} u_m \to \overline{\partial}^*_{\Lambda_{\tau,k}} u \tag{2.4}$$

in the respect $\Lambda_{\tau,k}$-weighted L^2-norms. □

Let $u \in \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap \Omega^{0,1}(\overline{\mathcal{M}_\varepsilon})$. We assume that u has compact support along t-direction. For $p \in \overline{\mathcal{M}_\varepsilon}$, there exists a neighborhood U_α of p in $\mathcal{M}'$ and orthonormal basis $\{\omega^j\}_{j=0}^2$ given as in (2.1) over U_α. By partition of unity, we assume that u has compact support in some $U_\alpha \cap \overline{\mathcal{M}_\varepsilon}$. Write $u = \sum_{j=0}^2 u_j \overline{\omega}^j$. Since u has compact support along t-direction and satisfies $\overline{\partial}$-Neumann boundary condition, so we have $u|_{Y_1} = 0$ and

$$\sum_{j=0}^{2} L_j(r) u_j = 0 \text{ along } \overline{Y_0} \setminus Y_1. \tag{2.5}$$

Write the volume form on Y_0 with respect to the Euclidean metric on some coordinate chart as ds. Also, in what follows, we write $O(A)$ for a quantity such that $|O(A)| \leq C|A|$ with C independent of the weight function (namely, independent of τ, k).

Theorem 2.3 (Basic estimate) *There exists τ, k sufficiently large such that for any $u \in \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap \Omega^{0,1}(\overline{\mathcal{M}_\uparrow})$ with u having compact support along t-direction, we have*

$$\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \gtrsim \|u\|^2_{\Lambda_{\tau,k}} + \sum_{\alpha,\beta=0}^{2} \int_{Y_0} r_{\alpha\overline{\beta}} u_\alpha \overline{u}_\beta e^{-\Lambda_{\tau,k}} h_0(z,t) e^{2\eta(t)} ds, \tag{2.6}$$

where $\{r_{\alpha\overline{\beta}}\}$ are given by $\partial\overline{\partial} r = \sum_{\alpha,\beta=0}^2 r_{\alpha\overline{\beta}} \omega^\alpha \wedge \overline{\omega}^\beta$. Moreover, For $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap L^2_{(0,1)}(\mathcal{M}_\uparrow, \Lambda_{\tau,k})$, we have

$$\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \gtrsim \|u\|^2_{\Lambda_{\tau,k}}. \tag{2.7}$$

Proof From Lemma 2.2, we only need prove the first part of this theorem. By partition of unity, we assume that supp $u \subset U_\alpha \cap \overline{\mathcal{M}_\varepsilon}$ for some neighborhood U_α. Let $\{\omega^j\}$ be the orthonormal basis defined in (2.1) over U_α.

$$\overline{\partial} u = \sum_{j<k} (\overline{L}_j u_k - \overline{L}_k u_j) \overline{\omega}^j \wedge \overline{\omega}^k + \text{lower order terms}. \tag{2.8}$$

Write $|||u|||^2_{\Lambda_{\tau,k}} = \sum_{\alpha,\beta=0}^2 \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} + \|u\|^2_{\Lambda_{\tau,k}}$. Then

$$\begin{aligned}\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} = {} & \sum_{\alpha,\beta=0}^{2} \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} - \sum_{\alpha,\beta=0}^{2} \int_{U_p} \overline{L}_\alpha(u_\beta) \overline{\overline{L}_\beta(u_\alpha)} e^{-\Lambda_{\tau,k}} h_0 e^{2\eta(t)} d_{Eucl} \\ & + O(|||u|||_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}).\end{aligned} \tag{2.9}$$

Let $v = \sum_{j=0}^{2} v_j \overline{\omega}^j$ and let $\chi \in C_0^\infty(U_\alpha \cap \mathcal{M}_\varepsilon)$. We have

$$\begin{aligned}\int_{U_p} \langle \overline{\partial}\chi, v\rangle e^{-\Lambda_{\tau,k}} h_0 e^{2\eta(t)} d_{Eucl} &= \sum_{j=0}^{2} \int_{U_p} \overline{L}_j(\chi)\overline{v}_j e^{-\Lambda_{\tau,k}} h_0 e^{2\eta(t)} d_{Eucl} \\ &= \sum_{j=0}^{2} \int_{U_p} \chi \overline{L}_j^*(\overline{v}_j \tilde{h})\tilde{h}^{-1}\tilde{h} d_{Eucl},\end{aligned} \tag{2.10}$$

where $\tilde{h} = e^{-\widetilde{\Lambda_{\tau,k}}} := e^{-\Lambda_{\tau,k}} h_0 e^{2\eta(t)}$ and $\overline{L}_j^*$ is the formal adjoint of $\overline{L}_j$ with respect to the Euclidean metric. Notice that $\overline{L}_j^* = -L_j + K_j$, with $K_j \in C^\infty(U_p)$ for all j. Since $u \in \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}})$, we have

$$\overline{\partial}^*_{\Lambda_{\tau,k}} u = -\sum_{j=0}^{2} \delta_j u_j + \text{lower order terms}$$

and

$$\|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} = \sum_{\alpha,\beta=0}^{2} (\delta_\alpha u_\alpha, \delta_\beta u_\beta)_{\Lambda_{\tau,k}} + O(\|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) + O(\|u\|^2_{\Lambda_{\tau,k}}), \tag{2.11}$$

where $(\cdot,\cdot)_{\Lambda_{\tau,k}}$ is the weighted inner product with respect to ds^2 and the weight function $\Lambda_{\tau,k}$. Here, $\delta_j u_j = e^{\widetilde{\Lambda_{\tau,k}}} L_j(e^{-\widetilde{\Lambda_{\tau,k}}} u_j)$. Combining (2.9) and (2.11), we have

$$\begin{aligned}&\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \\ &= \sum_{\alpha,\beta=0}^{2} \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} + \sum_{\alpha,\beta=0}^{2} (\delta_\alpha u_\alpha, \delta_\beta u_\beta)_{\Lambda_{\tau,k}} - \sum_{\alpha,\beta=0}^{2} (\overline{L}_\alpha u_\beta, \overline{L}_\beta u_\alpha)_{\Lambda_{\tau,k}} \\ &\quad + O(\|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) + O(\||u\||_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}).\end{aligned} \tag{2.12}$$

Since u satisfies $\overline{\partial}$-Neumann boundary condition, then integrating by parts

$$\begin{aligned}&(\delta_\alpha u_\alpha, \delta_\beta u_\beta)_{\Lambda_{\tau,k}} \\ &= -(\overline{L}_\beta \delta_\alpha u_\alpha, u_\beta)_{\Lambda_{\tau,k}} + O(\|\delta_\alpha u_\alpha\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) \\ &= ((\delta_\alpha \overline{L}_\beta - \overline{L}_\beta \delta_\alpha)u_\alpha, u_\beta)_{\Lambda_{\tau,k}} - (\delta_\alpha \overline{L}_\beta u_\alpha, u_\beta)_{\Lambda_{\tau,k}} + O(\|\delta_\alpha u_\alpha\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) \\ &= ((\delta_\alpha \overline{L}_\beta - \overline{L}_\beta \delta_\alpha)u_\alpha, u_\beta)_{\Lambda_{\tau,k}} + (\overline{L}_\beta u_\alpha, \overline{L}_\alpha u_\beta)_{\Lambda_{\tau,k}} - \int_{Y_0} L_\alpha(r)\overline{L}_\beta(u_\alpha)\overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} ds \\ &\quad + O(\||u\||_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}).\end{aligned} \tag{2.13}$$

Since u satisfies $\overline{\partial}$-Neumann boundary condition, it follows that $\sum_{\alpha=0}^{2} u_\alpha L_\alpha(r) = 0$ on $Y_0 \setminus Y_1$. Making use of Morrey's trick we have on $\overline{Y_0} \setminus Y_1$,

$$\sum_{\beta=0}^{2} \overline{u}_\beta \overline{L}_\beta \left(\sum_{\alpha=0}^{2} u_\alpha L_\alpha(r) \right) = 0,$$

that is,

$$-\sum_{\alpha,\beta=0}^{2} L_\alpha(r)\overline{L}_\beta(u_\alpha)\overline{u}_\beta = \sum_{\alpha,\beta=0}^{2} \overline{L}_\beta L_\alpha(r) u_\alpha \overline{u}_\beta. \tag{2.14}$$

Substituting (2.14) to (2.13), we have

$$\begin{aligned} &\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \\ &= \sum_{\alpha,\beta=0}^{2} \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} + \sum_{\alpha,\beta=0}^{2} ((\delta_\alpha \overline{L}_\beta - \overline{L}_\beta \delta_\alpha) u_\alpha, u_\beta)_{\Lambda_{\tau,k}} \\ &\quad + \sum_{\alpha,\beta=0}^{2} \int_{Y_0} \overline{L}_\beta L_\alpha(r) u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} ds \\ &\quad + O(\| |u\| |_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) + O(\|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}). \end{aligned} \tag{2.15}$$

By direct calculation,

$$\begin{aligned} ((\delta_\alpha \overline{L}_\beta - \overline{L}_\beta \delta_\alpha) u_\alpha, u_\beta)_{\Lambda_{\tau,k}} &= ((L_\alpha \overline{L}_\beta - \overline{L}_\beta L_\alpha) u_\alpha, u_\beta)_{\Lambda_{\tau,k}} \\ &\quad + ((\overline{L}_\beta L_\alpha \widetilde{\Lambda_{\tau,k}}) u_\alpha, u_\beta)_{\Lambda_{\tau,k}}. \end{aligned} \tag{2.16}$$

Write $\partial \omega^{\overline{\beta}} = C^{\overline{\beta}}_{t\overline{s}} w^t \wedge w^{\overline{s}}$ and $\overline{\partial} \omega^\beta = C^\beta_{t\overline{s}} \omega^t \wedge \omega^{\overline{s}}$, where both $C^{\overline{\beta}}_{t\overline{s}}, C^\beta_{t\overline{s}} \in C^\infty(U_p \cap \overline{\mathcal{M}_\varepsilon})$ and $C^\beta_{t\overline{s}} = -\overline{C^{\overline{\beta}}_{s\overline{t}}}$. For any smooth function g, write $\partial\overline{\partial} g = \sum_{\alpha,\beta=0}^{2} g_{\alpha\overline{\beta}} \omega^\alpha \wedge \omega^{\overline{\beta}}$. By direct calculation,

$$\partial\overline{\partial} g = [L_\alpha \overline{L}_\beta g + (\overline{L}_t g) C^{\overline{t}}_{\alpha,\overline{\beta}}] \omega^\alpha \wedge \omega^{\overline{\beta}}, \tag{2.17}$$

$$\overline{\partial}\partial g = [-\overline{L}_\beta L_\alpha g + (L_t g) C^t_{\alpha\overline{\beta}}] \omega^\alpha \wedge \overline{\omega}^\beta. \tag{2.18}$$

and

$$g_{\alpha\overline{\beta}} = L_\alpha \overline{L}_\beta g + (\overline{L}_t g) C^{\overline{t}}_{\alpha\overline{\beta}}. \tag{2.19}$$

The $\partial\overline{\partial} g + \overline{\partial}\partial g = 0$ implies that

$$L_\alpha \overline{L}_\beta g - \overline{L}_\beta L_\alpha g = (\overline{L}_t g) \overline{C^t_{\beta\overline{\alpha}}} - (L_t g) C^t_{\alpha\overline{\beta}}. \tag{2.20}$$

Substituting (2.20) to the first term on the right-hand side of (2.16), we have

$$((L_\alpha \overline{L}_\beta - \overline{L}_\beta L_\alpha)u_\alpha, u_\beta)_{\Lambda_{\tau,k}} = (\overline{C^r_{\beta\overline{\alpha}}}(\overline{L}_r u_\alpha) - C^t_{\alpha\overline{\beta}} L_t u_\alpha, u_\beta)_{\Lambda_{\tau,k}}. \tag{2.21}$$

Integrating by parts,

$$\begin{aligned} &-(C^t_{\alpha\overline{\beta}} L_t(u_\alpha), u_\beta)_{\Lambda_{\tau,k}} \\ &= -\int_{Y_0} C^t_{\alpha\overline{\beta}} L_t(r) u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} d_{Eucl} - \int_{U_p} L_t(\widetilde{\Lambda_{\tau,k}}) C^t_{\alpha\overline{\beta}} u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} d_{Eucl} \\ &\quad + \int_{U_p} L_t(C^t_{\alpha\overline{\beta}}) u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} d_{Eucl} + \int_{U_p} C^t_{\alpha\overline{\beta}} u_\alpha L_t(\overline{u}_\beta) e^{-\widetilde{\Lambda_{\tau,k}}} d_{Eucl} + O(\|u\|^2_{\Lambda_{\tau,k}}). \end{aligned} \tag{2.22}$$

Combining (2.16), (2.20), (2.22), (2.15) and using the notation (2.19), we have

$$\begin{aligned} &\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \\ &\quad = \sum_{\alpha,\beta=0}^{2} \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} + \sum_{\alpha,\beta=0}^{2} \int_{U_p} (\widetilde{\Lambda_{\tau,k}})_{\alpha\overline{\beta}} u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} d_{Eucl} \\ &\qquad + \sum_{\alpha,\beta=0}^{2} \int_{Y_0} r_{\alpha\overline{\beta}} u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} ds \\ &\qquad + O(\|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}) + O(\||u\||_{\Lambda_{\tau,k}} \cdot \|u\|_{\Lambda_{\tau,k}}). \end{aligned} \tag{2.23}$$

Recall

$$\widetilde{\Lambda_{\tau,k}} = (-\tau + 2)\log(\varepsilon^2 - |t|^2) + k\varphi(z,t) - \log h_0(z,t)$$

and $\partial\overline{\partial}\widetilde{\Lambda_{\tau,k}} = \sum_{\alpha,\beta=0}^{n} (\widetilde{\Lambda_{\tau,k}})_{\alpha\overline{\beta}} \omega^\alpha \wedge \omega^{\overline{\beta}}$, $(-\log(\varepsilon^2 - |t|^2))_{0\overline{0}} = 1$, then we can choose τ, k sufficiently large such that

$$((\widetilde{\Lambda_{\tau,k}})_{\alpha\overline{\beta}}) \geq k(\delta_{\alpha\beta}), \tag{2.24}$$

where $(\delta_{\alpha\beta})$ is the identity matrix. Then by big–small constants argument, we have

$$\begin{aligned} \|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} &\gtrsim \sum_{\alpha,\beta=0}^{2} \|\overline{L}_\alpha u_\beta\|^2_{\Lambda_{\tau,k}} + k\|u\|^2_{\Lambda_{\tau,k}} \\ &\quad + \sum_{\alpha,\beta=0}^{2} \int_{Y_0} r_{\alpha\overline{\beta}} u_\alpha \overline{u}_\beta e^{-\widetilde{\Lambda_{\tau,k}}} ds. \end{aligned} \tag{2.25}$$

Since Y_0 is strongly pseudoconvex, then the boundary term in (2.25) is positive and we have

$$\|\overline{\partial} u\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}} u\|^2_{\Lambda_{\tau,k}} \gtrsim k\|u\|^2_{\Lambda_{\tau,k}}. \tag{2.26}$$

Then by partition of unity and Lemma 2.2 we get the conclusion of the second part of Theorem 2.3. □

The basic estimate in Theorem 2.3 implies that there is no obstruction for solving the $\overline{\partial}$-equation on $\mathcal{M}_\varepsilon$. That is, if we denote by

$$H^q_{(2)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) := \frac{\text{Ker } \overline{\partial} : L^2_{(0,q)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) \to L^2_{(0,q+1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})}{\text{Im } \overline{\partial} : L^2_{(0,q-1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) \to L^2_{(0,q)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})}$$

the L^2 Dolbeault cohomology, then we have the following

Corollary 2.4 *For sufficiently large τ and k, when $q \geq 1$, $H^q_{(2)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) = 0$.*

Remark 2.5 The assumption that each fiber of the family $\pi : \overline{\mathcal{M}_\varepsilon} \to \Delta_\varepsilon$ is a stein manifold with strongly pseudoconvex boundary is crucial in this paper. Without this assumption there will be compact analytic variety in $\mathcal{M}_\varepsilon$ and we cannot find a strictly plurisubharmonic function as in Lemma 2.1 which play an important role as a weight function in establishing the L^2-estimate for the $\overline{\partial}$-operator on $\mathcal{M}_\varepsilon$. Without such weight function, the obstruction $H^q_{(2)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$, $q \geq 1$ of solving the $\overline{\partial}$-equation on $\mathcal{M}_\varepsilon$ may be an infinite dimensional space.

Let ω be a $\overline{\partial}$-closed $\Lambda_{\tau,k}$-weighted L^2-integrable $(0, 1)$-form. Then by Corollary 2.4, the $\overline{\partial}$-equation $\overline{\partial} u = \omega$ always has a unique solution $u \in L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$ with $u \perp \text{Ker}(\overline{\partial})$. Next, we will show that the $\overline{\partial}$-equation on $\mathcal{M}_\varepsilon$ has boundary regularity.

Theorem 2.6 *Let ω be a closed $\Lambda_{\tau,k}$-weighted L^2-integrable $(0, 1)$-form. Suppose $\omega \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$. There exists a unique $u \in L^2(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) \cap C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$ such that $\overline{\partial} u = \omega$ with $u \perp \text{Ker}(\overline{\partial})$.*

Proof Let $\overline{\partial} : L^2(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) \to L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$. Let $\overline{\partial}^*_{\Lambda_{\tau,k}}$ be its Hilbert adjoint. There exists a unique $u \in L^2(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$ such that $\overline{\partial} u = \omega$ with $u \perp \text{Ker}(\overline{\partial})$. By Theorem 2.3, both $\overline{\partial}$ and $\overline{\partial}^*_{\Lambda_{\tau,k}}$ have closed range in $L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$ and $L^2(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$, respectively. Then $u \perp \text{Ker}(\overline{\partial})$ implies that $u \perp \text{Rang}(\overline{\partial}^*_{\Lambda_{\tau,k}})^\perp$ and thus $u \in \text{Rang}(\overline{\partial}^*_{\Lambda_{\tau,k}})$. There exists a $\beta \in L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k})$ with $\beta \perp \text{Ker}(\overline{\partial}^*_{\Lambda_{\tau,k}})$ such that $\overline{\partial}^*_{\Lambda_{\tau,k}} \beta = u$. Hence $\overline{\partial} \beta = 0$ and thus $(\overline{\partial}\,\overline{\partial}^*_{\Lambda_{\tau,k}} + \overline{\partial}^*_{\Lambda_{\tau,k}} \overline{\partial})\beta = \omega$. By the localized version of Kohn's subelliptic estimate near Y_0 (see [12, Chapter 2]) and [12, Proposition 3.1.1, 3.1.11] we conclude that $\beta \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$. The proof of this conclusion is just a minor modification of the argument in the proof of the main theorem in [12, Main Theorem 2.1.7]. For the convenience of the readers, we include the necessary modification here. We will follow the notation of [12, Chapter 2]. We now define $Q(\phi, \phi) = \|\overline{\partial}\phi\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}}\phi\|^2_{\Lambda_{\tau,k}} + \|\phi\|^2_{\Lambda_{\tau,k}}$ for $\phi \in \mathcal{D}^{(0,1)}$, where $\mathcal{D}^{(0,1)}$ is now a space of $(0, 1)$-forms $\phi \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1) \cap \text{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}})$ with compact support in the t-direction. Q^δ is defined in the same way as in [12, pp. 32] with $\{\rho_j\}$ a partition of unity subordinate to a finite covering $\{U_{p_j}\}$ of $\overline{\mathcal{M}_\varepsilon}$. Let α be any $\Lambda_{\tau,k}$-weighted L^2-integrable $(0, 1)$-form. We assume that $\alpha \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$ and $\alpha = F(\phi)$ for some

$\phi \in \tilde{\mathcal{D}}^{(0,1)}$ where $\tilde{\mathcal{D}}^{(0,1)}$ is the completion of $\mathcal{D}^{(0,1)}$ under the $\Lambda_{\tau,k}$-weighted Q-norm. For any given point $p \in \overline{Y_0} \setminus Y_1$ and a neighborhood U_p of p in $\overline{\mathcal{M}_\varepsilon}$ with U_p not intersecting with Y_1, to see that the [12, Main Theorem 2.1.7 (2)] holds over U_p, we only need explain why $\xi\phi^{\delta_l}$ converges to $\xi\phi$ as $\delta_l \to 0^+$ for any $\xi \in C_0^\infty(U_p \cap \overline{\mathcal{M}_\varepsilon})$. Let $\{\eta_v\}$ be the sequence defined in the proof of Lemma 2.2. Substituting $\eta_v\phi^{\delta_l}$ to the (2.6), we have

$$\begin{aligned} &\|\overline{\partial}(\eta_v\phi^{\delta_l})\|^2_{\Lambda_{\tau,k}} + \|\overline{\partial}^*_{\Lambda_{\tau,k}}(\eta_v\phi^{\delta_l})\|^2_{\Lambda_{\tau,k}} \\ &\quad \gtrsim \|\eta_v\phi^{\delta_l}\|^2_{\Lambda_{\tau,k}} + C_v \int_{Y_0} |\eta_v\phi^{\delta_l}|^2 e^{-\Lambda_{\tau,k}} h_0 e^{2\eta(t)} ds, \end{aligned} \tag{2.27}$$

where C_v is a constant which does not depend on ϕ^{δ_l}. Then as in [12, pp. 45], there is a constant $C_{v,s}$ such that

$$\|\eta_v\phi^{\delta_l}\|_s \le C_{v,s} \text{ uniformly as } \delta_l \to 0 \tag{2.28}$$

with $s \ge 1$, $v \gg 1$. Then by a diagonal selecting process and making use of Rellich lemma, we can assume that ϕ^{δ_l} converges to ϕ_0 in the Sobolev $\|\cdot\|_s$-norm for each $s \ge 1$, $v \gg 1$ over any compact subset of $\overline{\mathcal{M}_\varepsilon} \setminus Y_1$. Thus, by Sobolev embedding theorem $\phi_0 \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$. Notice that $Q(\phi^{\delta_l}, \phi^{\delta_l}) \le Q^{\delta_l}(\phi^{\delta_l}, \phi^{\delta_l}) = (\alpha, \phi^{\delta_l})$, where $\alpha = F(\phi) = F^{\delta_l}(\phi^{\delta_l})$. By a big–small constant argument, it follows that

$$\|\overline{\partial}\phi^{\delta_l}\|_{\Lambda_{\tau,k}} \lesssim \|\alpha\|_{\Lambda_{\tau,k}}, \|\overline{\partial}^*_{\Lambda_{\tau,k}}\phi^{\delta_l}\|_{\Lambda_{\tau,k}} \lesssim \|\alpha\|_{\Lambda_{\tau,k}}, \|\phi^{\delta_l}\|_{\Lambda_{\tau,k}} \lesssim \|\alpha\|_{\Lambda_{\tau,k}}. \tag{2.29}$$

An immediate consequence of (2.29) is that $\phi_0 \in L^2_{(0,1)}(\mathcal{M}_\varepsilon, \Lambda_{\tau,k}) \cap \mathrm{Dom}(\overline{\partial}^*_{\Lambda_{\tau,k}}) \cap \mathrm{Dom}(\overline{\partial})$. Obviously, $\eta_v\phi_0 \in \mathcal{D}^{(0,1)}$ for all $v \gg 1$ and the limit of $\{\eta_v\phi_0\}$ with respect to the $\Lambda_{\tau,k}$-weighted Q-norm is ϕ_0, thus $\phi_0 \in \tilde{D}^{(0,1)}$. Next, for any $\psi \in \mathcal{D}^{(0,1)}$,

$$Q(\phi, \psi) = (\alpha, \psi) = Q^{\delta_l}(\phi^{\delta_l}, \psi) = Q(\phi^{\delta_l}, \psi) + O(\delta_l C_{v,1}\|\psi\|_1) \tag{2.30}$$

for some $v \gg 1$ where $C_{v,1}$ is the constant given in (2.28). Letting $\delta_l \to 0$, we have

$$Q(\phi - \phi_0, \psi) = 0, \forall \psi \in \mathcal{D}^{(0,1)}. \tag{2.31}$$

Thus, $\phi = \phi_0$. After [12, Main theorem 2.1.7] is modified to $\mathcal{M}_\varepsilon$, [12, Propositions 3.1.1, 3.1.11] need no change at all for deriving the smoothness of our β to a small neighborhood of p in $\overline{\mathcal{M}_\varepsilon} \setminus Y_1$. □

3 Simultaneous Embedding of CR Manifolds

Using what we have established in Sect. 2, we can now give a proof of the Theorem 1.6. The key step is to prove the following global extension theorem.

Theorem 3.1 *Let f be a holomorphic function on M_0 which is smooth up to the boundary X_0. Then f admits a holomorphic extension $\hat{f}$ on $\mathcal{M}_\varepsilon$ which is smooth over $\overline{\mathcal{M}_\varepsilon} \setminus Y_1$.*

Proof By a Lemma of Bell [3, Sect. 4], we can find a finite open covering $\{U_\alpha\}$ of $\mathcal{M}_\varepsilon$ such that each $(U_\alpha \subset \mathcal{M})$ is a strongly pseudoconvex manifold with connected smooth boundary and the intersection $(\partial U_\alpha \cap \partial \mathcal{M})$ (if not empty) contains an open subset of the strongly pseudoconvex boundary of $(\mathcal{M})$. Suppose ε is small and we assume that $U_\alpha \cap X_0 \neq \emptyset$ for all αs. We denote by f_α the restriction of f on $U_\alpha \cap X_0$ for each α. Then each f_α is a holomorphic function which is smooth up to the boundary of $U_\alpha \cap X_0$. By [1, Theorem 1], f_α can be holomorphically extended to U_α and if we denote the extension by $\tilde{f}_\alpha$ then $\tilde{f}_\alpha$ is smooth up to the boundary of U_α. Choose a partition of unity $\{\chi_\alpha\}$ subordinate to the covering $\{U_\alpha\}$. Put $\tilde{f} = \sum_\alpha \chi_\alpha \tilde{f}_\alpha$. Choose a cut-off function $\chi(t) \in C_0^\infty(\Delta_\varepsilon)$ satisfying $\chi \equiv 1$, when$|t| \leq \frac{\varepsilon}{2}$. Set $\omega = \frac{1}{t}\overline{\partial}(\chi(t)\tilde{f})$. Then by the same argument as in [18, pp. 364] we have that $\omega \in \Omega^{0,1}(\overline{\mathcal{M}_\varepsilon})$ and ω has compact support along t-direction. By Theorem 2.6, there exists $u \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$ such that $\overline{\partial} u = \omega$. Thus, $\overline{\partial}(\chi(t)\tilde{f} - tu) = 0$. Write $\hat{f} = \chi(t)\tilde{f} - tu$. Then $\hat{f}$ is a holomorphic function on $\mathcal{M}_\varepsilon$ and $\hat{f} \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$, $\hat{f}|_{X_0} = f$. □

3.1 Proof of Theorem 1.6

Proof Let $F_0 = (f_1, \cdots, f_m) : X_0 \to \mathbb{C}^m$ be a smooth CR embedding. Then by Bochner extension each f_j can be holomorphically extended to M_0 which is still denoted by f_j and f_j is smooth up to the boundary X_0. By Theorem 3.1, each f_j admits a holomorphic extension $\hat{f}_j$ to $\mathcal{M}_\varepsilon$. Moreover, each $\hat{f}_j \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$ for $1 \leq j \leq m$. Set $\hat{G} = (\hat{f}_1, \cdots, \hat{f}_m) : \mathcal{M}_\varepsilon \to \mathbb{C}^m$. Then $\hat{G}$ is a holomorphic map and $\hat{G} \in C^\infty(\overline{\mathcal{M}_\varepsilon} \setminus Y_1)$ with $\hat{G}|_{X_0} = F_0$. Set $G = (\hat{G}, \pi) : \mathcal{X}_\varepsilon \to \mathbb{C}^{m+1}$ with $G(p) = (\hat{G}(p), \pi(p)) \; \forall \, p \in \mathcal{X}_\varepsilon$. Then G is a CR embedding when ε is sufficiently small and $G|_{X_t} : X_t \to \mathbb{C}^m \times \{t\}$ is a CR embedding. $G|_{X_0} = (F_0, 0)$. □

The arguments in this work give some partial result of Huang's problem [19] under the condition each fiber bounds a stein space without singularities. It does not work on the cases when each fiber bounds a strongly pseudoconvex complex manifold which have compact analytic varieties in the interior. Motivated by Huang–Luk–Yau's work [18], we state the following open problem:

Problem 3.2 *Let $\{X_t\}_{t \in \Delta}$ be a CR family of 3-dimensional strongly pseudoconvex CR manifolds. Suppose the total space $X = \cup_{t \in \Delta} X_t$ bounds a holomorphic family of complex manifolds $\{M_t\}_{t \in \Delta}$. Suppose that*

$$\dim H^1_{(2)}(M_t) \equiv constant > 0,$$

where $H^1_{(2)}(M_t) = \frac{\mathrm{Ker}\overline{\partial}: L^2_{(0,1)}(M_t) \to L^2_{(0,2)}(M_t)}{\mathrm{Im}\overline{\partial}: L^2(M_t) \to L^2_{(0,1)}(M_t)}$. Assume that X_0 can be CR embedded into some $\mathbb{C}^N$. Can the nearby X_t be CR embedded into the same $\mathbb{C}^N$ with the embedding maps CR depending on the parameters?

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